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ON THE ACCURACY OF CONSERVATION OF THE ADIABATIC INVARIANT*

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A Hamiltonian system with one degree of freedom is considered. The system depends on the parameter ξ which varies slowly with time t: $\xi = \xi(\varepsilon t)$, $0 < \varepsilon \ll 1$ and tends in a sufficiently regular manner to well defined limits as $t \to \pm\infty$. When $t \to \pm\infty$, the adjabatic invariant, acting along the trajectory of such system, has the limit meanings $I_\pm$. Their difference $\Delta I = I_+ - I_-$ is estimated.

The problem of estimating ΔI appears in classical mechanics /1,2/, quantum mechanics /3/ and in the theory of waveguides /4/. In the case when the dependence of ξ on εt is finite ($\xi(\varepsilon t) = \text{const}$ for sufficiently large εt) and infinitely differentiable, the author of /5/ shows that ΔI decreases, as $\varepsilon \to 0$, faster than any power of ε. For the linear systems with the frequency depending analytically on εt, the known asymptotic of ΔI is found to be exponential $\Delta I = O(\exp(-c/\varepsilon))$, $c = \text{const}$ /3,6/. An erroneous proof of the exponential smallness of ΔI in given in /1/ for the case of an analytic $\xi(\varepsilon t)$ for the general nonlinear systems.

Below the problem of estimating ΔI is considered with help of the perturbations procedure in the action-angle variables. For the case when the dependence of ξ on εt is finite and has a finite smoothness, a power asymptotics is obtained for ΔI and the exponential smallness of ΔI in ε is proved for the case of analytic $\xi(\varepsilon t)$.

1. Equations in the action-angle variables. The Hamiltonian of the problem in question has the form

$$E = E(p, q, \xi), \quad \xi = \xi(\lambda), \quad \lambda = \varepsilon t \tag{1.1}$$

where p and q denote the canonical variables. We assume that E is an analytic function of p, q and ξ.

Let a region filled with closed trajectories exist for every ξ on the phase plane of the unperturbed ($\xi = \text{const}$) problem. The action-angle variables of the unperturbed problem are defined in this region /1/. The action $I = I(p, q, \xi)$ is an area divided by 2π, bounded by an unperturbed trajectory passing through the point (p, q). The angle $\varphi = \varphi(p, q, \xi) \bmod 2\pi$ is an angular coordinate along the trajectory, varying uniformly in the unperturbed system. The variable change $p, q \to I, \varphi$ is canonical and time-dependent. The change of the variables I, φ in the initial problem is described by a Hamiltonian system with the Hamiltonian

$$H(I, \varphi, \lambda) = H_0(I, \lambda) + \varepsilon H_1(I, \varphi, \lambda) \tag{1.2}$$

where H_0 is the original Hamiltonian of E expressed in terms of I and λ

$$H_1 = (d\xi/d\lambda)\, G(I, \varphi, \lambda) \tag{1.3}$$
$$G = \frac{1}{\omega}\left[-\int_0^\varphi \frac{\partial E}{\partial \xi}\, d\varphi + \frac{\varphi}{2\pi}\int_0^{2\pi} \frac{\partial E}{\partial \xi}\, d\varphi\right]$$
$$\omega = \omega(I, \lambda) = \partial H_0/dI$$

and H is an analytic function of I and φ, and 2π-periodic in φ.

We shall use the angle brackets $\langle\cdot\rangle^\varphi$ for the 2π-periodic functions of φ to denote the averaging over φ, and the curly brackets $\{\cdot\}^\varphi$ for the purely periodic part: $\{\cdot\}^\varphi = (\cdot) - \langle\cdot\rangle^\varphi$. Further, in order to reduce the amount of notations used we shall assume that $\langle H_1\rangle^\varphi = 0$, otherwise we shall have to replace, in the arguments that follow, H_0 by $H_0 + \varepsilon\langle H_1\rangle^\varphi$, and H_1 by $\{H_1\}^\varphi$.

2. The procedure of the perturbation theory. The procedure described here of asymptotic integration of the system with the Hamiltonian (1.2), analogous to that given in /7/, will be used to obtain the estimates of variation in the value of the adiabatic invariant.

Let us carry out, in the system with the Hamiltonian (1.2), a canonical, nearly identical variable transformation with a so far undefined generating function $W = J\varphi + \varepsilon S(J, \varphi, \lambda)$. Here J and ψ are the new canonical variables connected with I and φ by the relations

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$$I = J + \varepsilon \frac{\partial S}{\partial \varphi}, \quad \psi = \varphi + \varepsilon \frac{\partial S}{\partial J} \tag{2.1}$$

The variation in J and ψ is described by a Hamiltonian system with the Hamiltonian

$$\Phi(J, \psi, \lambda) = \varepsilon^2 \frac{\partial S}{\partial \lambda} + H_0\left(J + \varepsilon \frac{\partial S}{\partial \varphi}\right) + \varepsilon H_1\left(J + \varepsilon \frac{\partial S}{\partial \varphi}, \varphi, \lambda\right) \tag{2.2}$$

In the right-hand part of (2.2) φ must be expressed in terms of J and ψ in accordance with (2.1). Let us choose

$$S = \frac{1}{\partial H_0 / \partial I} \left\{ \int_0^\varphi H_1 d\varphi \right\}^\varphi$$

Since $\langle H_1 \rangle^\varphi = 0$, the above expression is correct and S is a 2π-periodic function of φ with a zero mean value. For S chosen in such a manner we have

$$\begin{gathered} \Phi = H_0(J, \lambda) + \varepsilon^2 F \\ \varepsilon^2 F = \varepsilon^2 \frac{\partial S}{\partial \lambda} + \left(H_0\left(J + \varepsilon \frac{\partial S}{\partial \varphi}, \lambda\right) - H_0(J, \lambda) - \varepsilon \frac{\partial H_0(J, \lambda)}{\partial J} \frac{\partial S}{\partial \varphi}\right) + \\ \varepsilon\left(H_1\left(J + \varepsilon \frac{\partial S}{\partial \varphi}, \varphi, \lambda\right) - H_1(J, \varphi, \lambda)\right), \quad F = O(1) \end{gathered} \tag{2.3}$$

Let us denote

$$\Phi_0 = H_0 + \varepsilon^2 \langle F \rangle^\psi, \quad \Phi_1 = \{F\}^\psi$$

Then the Hamiltonian will become

$$\Phi = \Phi_0(J, \lambda) + \varepsilon^2 \Phi_1(J, \psi, \lambda)$$

which is completely analogous to (1.2) except that the term depending on the phase ψ is now of the order of ε^2. Repeating the above procedure once more, we obtain a Hamiltonian in which the phase dependence appears only in the terms of the order of ε^3. Finally, after n steps of the above procedure we obtain the Hamiltonian in the form

$$\Phi(J, \psi, \lambda) = \Phi_0(J, \lambda) + \varepsilon^{n+1} \Phi_1(J, \psi, \lambda) \tag{2.4}$$

where the new variables and the Hamiltonian are still expressed in terms of J, ψ and Φ.

Neglecting in the Hamiltonian (2.4) the last term, we obtain the following relations for the variation of J and ψ with time:

$$J = \text{const}, \quad \psi = \psi_0 + \frac{1}{\varepsilon} \int_{\lambda_0}^{\lambda} \frac{\partial \Phi_0(J, \nu)}{\partial J} d\nu, \quad \psi_0 = \text{const} \tag{2.5}$$

Analysis of the corrections brought in by each consecutive step of the procedure shows that the relations (2.5) describe the variation in J with the accuracy of $O(\varepsilon^{n+1})$ and in ψ with the accuracy of $O(\varepsilon^n)$ on the time intervals of the order $1/\varepsilon$ (provided that the step in question is feasible). From (2.3) we see that every step of the procedure reduces the smoothness of the Hamiltonian with respect to λ, by 1. It follows therefore that the number of steps which can be carried out depends on the smoothness H of the Hamiltonian with respect to λ.

3. The asymptotics of variation of the adiabatic invariant for a finite perturbation of finite smoothness. Let the perturbation be finite $\xi(\lambda) \equiv \text{const}$ for $\lambda < \lambda_-$ and $\lambda > \lambda_+$, where $\lambda_\pm$ are constant. Let the first $n-1$ derivatives of $\xi(\lambda)$ be continuous (and hence vanish) at the point $\lambda_\pm$, and the n-th derivative be discontinuous, i.e. $n \geqslant 1$. Let the function $\xi(\lambda)$ be differentiable $n+2$ times for $\lambda \in (\lambda_-, \lambda_+)$ and its derivatives bounded. We shall calculate the asymptotics of variation of the adiabatic invariant.

Let us denote by $I(\lambda)$, $\varphi(\lambda)$ the solution of a system with the Hamiltonian (1.2) and $I_\pm = I(\lambda_\pm)$, $\varphi_- = \varphi(\lambda_-)$, $\Delta I = I_+ - I_-$. We shall write, for a 2π-periodic function $f(\varphi)$ with zero mean value

$$(Lf)(\varphi) = -\left\{ \int_0^\varphi f(\gamma) d\gamma \right\}^\varphi$$

Theorem 1. The quantity ΔI has the following asymptotics

$$\Delta I = \varepsilon^n (R(\lambda_+ - 0) - R(\lambda_- + 0)) + O(\varepsilon^{n+1}), \quad R(\lambda) = -\frac{\xi^{(n)}(\lambda)}{\omega^n(I_-, \lambda)} (L^{n-1}G)(I_-, \varphi_*(\lambda), \lambda) \tag{3.1}$$

where L^{n-1} denotes the operator L raised to the $(n-1)$-th power, G is given by (1.3), and

$$\varphi_*(\lambda) = \varphi_- + \varepsilon^{-1} \int_{\lambda_-}^{\lambda} \omega\left(I_- + \varepsilon \frac{H_1(I_-, \varphi_-, \lambda_- + 0)}{\omega(I_-, \lambda_-)}, \nu\right) d\nu$$

Proof. When $\lambda \in (\lambda_-, \lambda_+)$, the smoothness allows $n+1$ steps of the procedure given in Sect.2. Let us denote by (J, ψ) the variables introduced in the n-th step, and by $W = J\varphi + \varepsilon S(J, \varphi, \lambda)$ the generating function of the substitution $(I, \varphi) \to (J, \psi)$. Let $J(\lambda), \psi(\lambda)$ denote the solution $I(\lambda), \varphi(\lambda)$ in terms of the variables (J, ψ). From the formulas for the substitution of variables it follows that

$$I(\lambda_+) - I(\lambda_-) = J(\lambda_+ - 0) - J(\lambda_- + 0) + \varepsilon \frac{\partial S(J(\lambda), \varphi(\lambda), \lambda)}{\partial \varphi} \bigg|_{\lambda_- + 0}^{\lambda_+ - 0} \tag{3.2}$$

and the assertion of Sect.2 concerning the accuracy of (2.5) yields

$$J(\lambda_+ - 0) - J(\lambda_- + 0) = O(\varepsilon^{n+1})$$

Analysis of the procedure of Sect.2 shows that

$$S(I, \varphi, \lambda_\pm \mp 0) = \frac{\varepsilon^{n-1}}{\omega^n(I, \lambda_\pm)} \left(\frac{\partial^{n-1}}{\partial \lambda^{n-1}} L^n H_1\right)_{\lambda = \lambda_\pm \mp 0}$$

Let us now substitute the above expression into the right-hand part of (3.2) and replace in S $J(\lambda)$ by I_- and $\varphi(\lambda_+)$ by $\varphi_*(\lambda_+)$. The admissible error in this case is $O(\varepsilon^{n+1})$. Taking into account the expression for H_1 (1.3), we obtain the required expression (3.1).

Note. The formula (3.1) can be rewritten in the form

$$\Delta I = -\int_{\lambda_-}^{\lambda_+} \frac{\partial H_1(I_-, \varphi_*(\lambda), \lambda)}{\partial \varphi} d\lambda + O(\varepsilon^{n+1})$$

since the asymptotics of the integral is given here by the right-hand part of (3.1).

4. Exponential estimate of the variation of the adiabatic invariant for an analytic perturbation. Let now the function $\xi(\lambda)$ be analytic in some strip $|\mathrm{Im}\,\lambda| < \rho$. Let also the integral

$$\int_{-\infty}^{\infty} \left|\frac{d\xi}{d\lambda}\right| d\lambda \tag{4.1}$$

be uniformly bounded in this strip for the straight lines $\mathrm{Im}\,\lambda = \mathrm{const}$ and let $d\xi/d\lambda \to 0$ as $\mathrm{Re}\,\lambda \to \pm\infty$. We shall assume, as before, that the initial Hamiltonian $E(p, q, \xi)$ is an analytic function of its arguments, so that the passage to the action-angle variables yields the Hamiltonian H (1.2) analytic in the complex domain

$$|\mathrm{Im}\,\lambda| < \rho, \quad |I - I_0| < \varkappa, \quad |\mathrm{Im}\,\varphi| < \sigma$$

where $\varkappa > 0, \sigma > 0, I_0$ are constants and I_0 is real.

Let $I(\lambda), \varphi(\lambda)$ be the solution of the Hamiltonian system in question, and $I(0) = I_0$. The convergence of the integral (4.1) implies the existence of

$$I_\pm = \lim_{\lambda \to \pm\infty} I(\lambda), \quad \Delta I = I_+ - I_-$$

Theorem 2. The variation of the adiabatic invariant ΔI is exponentially small: $\Delta I = O(\exp(-c_1/\varepsilon))$, $c_1 > 0$ is a constant.

Proof. This is based on the procedure of Sect.3 and uses the technique of estimates given in /7,8/. After a large number of steps $(\sim 1/\varepsilon)$ of the procedure, the dependence of the Hamiltonian on phase concentrates in the exponentially small terms, and this implies the exponential smallness of ΔI.

Let us pass to the estimates. We denote by $\rho_0, \varkappa_0, \sigma_0, \vartheta, \Theta, M_0$ the positive constants such, that when

$$|\operatorname{Im}\lambda| < \rho_0, \quad |I - I_0| < \varkappa_0, \quad |\operatorname{Im}\varphi| < \sigma_0$$

then the inequalities

$$\vartheta < |\partial H_0/\partial I| < \Theta, \quad |H_1| < M_0 \tag{4.2}$$

hold.

Consider the procedure of Sect.2. Let us carry out $N \geqslant 0$ steps of this procedure. After the k-th step the Hamiltonian is reduced to the form

$$\Phi^{(k)}(I, \varphi, \lambda) = \Phi_0^{(k)}(I, \lambda) + \varepsilon\Phi_1^{(k)}(I, \varphi, \lambda), \quad \langle\Phi_1^{(k)}\rangle^\varphi = 0, \quad |\Phi_1^{(k)}| < M_k = M_k(\varepsilon)$$

in the region

$$|\operatorname{Im}\lambda| < \rho_k, \quad |I - I_0| < \varkappa_k, \quad |\operatorname{Im}\varphi| < \sigma_k \tag{4.3}$$

$$\rho_k = \rho_0 - k\Delta, \quad \varkappa_k = \varkappa_0 - 4\sum_{i=1}^{k}\delta_i, \quad \sigma_k = \sigma_0 - 4\sum_{i=1}^{k}\beta_i$$

$$\delta_i = 2^{-i}\delta, \quad \beta_i = 2^{-i}\beta, \quad \delta = {}^1/_8\varkappa_0, \quad \beta = {}^1/_8\sigma_0$$

We shall show that be choosing suitably $\Delta = \Delta(\varepsilon)$, $M_i = M_i(\varepsilon)$ we can obtain $N \sim 1/\varepsilon$ such, that the following estimates will hold at all steps of the procedure:

$${}^1/_2\vartheta < |\partial\Phi_0^{(i)}/\partial I| < 2\Theta \tag{4.4}$$

$$\frac{\varepsilon M_i}{\delta_{i+1}} < \frac{\vartheta}{2}, \quad \frac{\varepsilon M_i}{\delta_{i+1}\beta_{i+1}} < \frac{\vartheta}{\pi}, \quad \frac{\Delta M_i}{\delta_{i+1}} < \frac{\pi\vartheta}{2\Theta} \tag{4.5}$$

The proof is by induction. The estimates (4.4) and (4.5) clearly hold for the initial Hamiltonian. Assume that they hold for k steps and, that $\rho_{k+1} = \rho_k - \Delta > 0$. Consider now the $(k+1)$-th step, denoting the new variables by J and ψ. The generating function of the variable change W has the form

$$W = J\varphi + \varepsilon S(J, \varphi, \lambda), \quad \frac{\partial S}{\partial\varphi} = -\frac{\Phi_1^{(k)}}{\partial\Phi_0^{(k)}/\partial I}, \quad \langle S\rangle^\varphi = 0 \tag{4.6}$$

Using the assumptions (4.4) and (4.5) we can show that when

$$|I - I_0| < \varkappa_k - 2\delta_{k+1}, \quad |\operatorname{Im}\varphi| < \sigma_k - 2\beta_{k+1}$$

the change governed by the generating function W is well defined and the domain of variation of the new variables J, ψ contains the region

$$|J - I_0| < \varkappa_k - 3\delta_{k+1}, \quad |\operatorname{Im}\psi| < \sigma_k - 3\beta_{k+1}$$

The proof duplicates the known argument of (/7/, Sect.4) and is therefore omitted.

Let us obtain the estimates for the new Hamiltonian. Using the Cauchy inequalities /7/, the definition of S (4.6) and the estimates (4.4), (4.5) we find that when

$$|\operatorname{Im}\lambda| < \rho_k - \Delta, \quad |I - I_0| < \varkappa_k - 2\delta_{k+1}, \quad |\operatorname{Im}\varphi| < \sigma_k$$

then the inequalities

$$\left|\frac{\partial\Phi_1^{(k)}}{\partial I}\right| < \frac{M_k}{2\delta_{k+1}}, \quad \left|\frac{\partial^2\Phi_0^{(k)}}{\partial I^2}\right| < \frac{\Theta}{\delta_{k+1}}, \quad \left|\frac{\partial S}{\partial\varphi}\right| < \frac{2M_k}{\vartheta}, \quad |S| < \frac{2\pi M_k}{\vartheta}, \quad \left|\frac{\partial S}{\partial\lambda}\right| < \frac{2\pi M_k}{\Delta\vartheta} \tag{4.7}$$

hold. The new Hamiltonian is computed according to (2.3). Estimating the right-hand part of (2.3) in the region

$$|\operatorname{Im}\lambda| < \rho_k - \Delta, \; |J - I_0| < \varkappa_k - 3\delta_{k+1}, \quad |\operatorname{Im}\varphi| < \sigma_k$$

we find, with the help of (4.4), (4.5) and (4.7), that

$$|\varepsilon F| < \varepsilon\left(\frac{2\pi M_k}{\Delta\vartheta} + \frac{M_k^2}{\delta_{k+1}\vartheta} + \frac{2M_k^2\Theta}{\delta_{k+1}\vartheta^2}\right) = \frac{2\pi\varepsilon M_k}{\Delta\vartheta}\left(1 + \frac{\Delta M_k}{2\pi\delta_{k+1}} + \frac{\Delta M_k\Theta}{\pi\delta_{k+1}\vartheta}\right) < \frac{4\pi\varepsilon M_k}{\Delta\vartheta}$$

and this yields

$$|\Phi_1^{(k+1)}| = |\{\varepsilon F\}^\psi| < \frac{8\pi\varepsilon M_k}{\Delta\vartheta} \tag{4.8}$$

$$|\Phi_0^{(k)} - \Phi_0^{(k+1)}| = |\langle \varepsilon^2 F\rangle^{\psi}| < \frac{4\pi\varepsilon^2 M_k}{\Delta\vartheta} \tag{4.9}$$

Expression (4.9) and the Cauchy inequality together show that when $|J - I_0| < \varkappa_k - 4\delta_{k+1}$ then

$$\left|\frac{\partial\Phi_0^{(k)}}{\partial J} - \frac{\partial\Phi_0^{(k+1)}}{\partial J}\right| < \frac{4\pi\varepsilon^2 M_k}{\Delta\vartheta\delta_{k+1}} \tag{4.10}$$

Let us write

$$\Delta = \frac{32\pi}{\vartheta}\varepsilon, \quad M_{k+1} = \frac{8\pi\varepsilon}{\Delta\vartheta} M_k = {}^1/_4 M_k \tag{4.11}$$

Then

$$M_i = \left(\frac{1}{4}\right)^i M_0, \quad \frac{M_i}{\delta_{i+1}} = \left(\frac{1}{2}\right)^{i-1}\frac{M_0}{\delta}, \quad \frac{M_i}{\delta_{i+1}\beta_{i+1}} = \frac{4M_0}{\delta\beta}, \quad 0 \leqslant i \leqslant k+1 \tag{4.12}$$

and from this it is clear that the relations (4.5) remain valid for the $(k+1)$-th step of the procedure. Further, from (4.2) and (4.10)—(4.12) we obtain

$$\left|\frac{\partial\Phi_0^{(k+1)}}{\partial J}\right| > \vartheta - {}^1/_4\varepsilon\sum_{i=0}^{k}\left(\frac{1}{2}\right)^i\frac{M_0}{\delta} > \vartheta - \frac{\varepsilon M_0}{2\delta}, \qquad \left|\frac{\partial\Phi_0^{(k+1)}}{\partial J}\right| < \Theta + \frac{\varepsilon M_0}{2\delta}$$

Since $\delta = {}^1/_8\varkappa = \mathrm{const}$ it follows that for a sufficiently small ε the conditions (4.4) hold also for the $(k+1)$-th step.

In this manner we can repeat the variable change as long as $\rho_k > 0$. This means that we can carry out

$$N = \left\lfloor\frac{\rho_0}{\delta}\right\rfloor \geqslant \frac{\rho_0\vartheta}{32\pi\varepsilon} - 1$$

steps. After the N-th step we obtain

$$|\Phi_1^{(N)}| < M_0\left(\frac{1}{4}\right)^N = O(\exp(-c_1/\varepsilon))$$

i.e. $\Phi_1^{(N)}$ is an exponentially small quantity.

We shall show that the function $\Phi_1^{(N)}$ is also exponentially small on the integral norm

$$\|\Phi_1^{(k)}\| = \sup_{|\mathrm{Im}\lambda| < \rho_k}\int_{-\infty}^{\infty}\sup_{I,\varphi}|\Phi_1^{(k)}|\,d\lambda$$

where the upper bound of the integrand is taken over all I and φ belonging to the domain of definition of $\Phi_1^{(k)}$ (4.2), the integral is taken along the straight line $\mathrm{Im}\,\lambda = \mathrm{const}$, and the upper bound preceding the integral is taken over all such lines with $|\mathrm{Im}\,\lambda| < \rho_k$.

Let us return to the $(k+1)$-th step of the procedure of Sect.2. We shall obtain the estimate for the integral norm of the function $\partial S/\partial\lambda$ in the strip $|\mathrm{Im}\,\lambda| < \rho_{k+1} = \rho_k - \Delta$ in terms of the integral norm of S in the strip $|\mathrm{Im}\,\lambda| < \rho_k$. In order to simplify the formula, we shall write, for the time being, $S(\lambda) = S(I, \varphi, \lambda)$. By virtue of the Cauchy integral formula we have

$$\frac{\partial S}{\partial\lambda} = \frac{1}{2\pi i}\oint_{\Gamma}\frac{S(\lambda+\zeta)}{\zeta^2}\,d\zeta$$

where the contour Γ is a circle of radius Δ with the center at the point 0. Therefore

$$\left\|\frac{\partial S}{\partial\lambda}\right\| = \frac{1}{2\pi}\sup_{|\mathrm{Im}\lambda| < \rho_{k+1}}\int_{-\infty}^{\infty}\sup_{I,\varphi}\left|\oint_{\Gamma}\frac{S(\lambda+\zeta)}{\zeta^2}\,d\zeta\right|d\lambda$$

Changing the order of integration, we obtain

$$\left\|\frac{\partial S}{\partial\lambda}\right\| \leqslant \frac{1}{2\pi}\oint_{\Gamma}\left(\sup_{|\mathrm{Im}\lambda| < \rho_{k+1}}\int_{-\infty}^{\infty}\sup_{I,\varphi}|S(\lambda+\zeta)|\,d\lambda\right)\frac{|d\zeta|}{|\zeta|^2} = \frac{\|S\|}{2\pi}\oint_{\Gamma}\frac{|d\zeta|}{|\zeta|^2} = \frac{\|S\|}{\Delta}$$

Estimating now $\|S\|$ is terms of $\|\Phi_1^{(k)}\|$ and using (4.4), (4.6) we find, that at the

$(k+1)$-th step of the procedure

$$\left\|\frac{\partial S}{\partial \lambda}\right\| \leqslant \frac{2\pi}{\Delta\theta}\|\Phi_1^{(k)}\| \tag{4.13}$$

Finally, estimating $\|\Phi_1^{(k+1)}\|$ and using (2.3), (4.13), (4.4), (4.5), (4.7) and (4.11), we obtain

$$\|\Phi_1^{(k+1)}\| \leqslant {}^1/_4\|\Phi_1^{(k)}\|$$

Therefore

$$\|\Phi_1^{(N)}\| \leqslant \left(\frac{1}{4}\right)^N \|H_1\| = O\left(\exp\left(-\frac{c_1}{\varepsilon}\right)\right) \sup_{|\mathrm{Im}\,\lambda|<\rho} \int_{-\infty}^{\infty} \left|\frac{d\xi}{d\lambda}\right| d\lambda = O\left(\exp\left(-\frac{c_1}{\varepsilon}\right)\right)$$

Q.E.D.

Let us now study the variation of the adiabatic invariant over an infinite period. Let us denote by J, ψ the variables introduced at the N-th step of the procedure. The domain of variation of these variables contains the region

$$|J - I_0| < {}^1/_2\varkappa_0, \quad |\mathrm{Im}\,\psi| < {}^1/_2\sigma_0$$

Let $J(\lambda), \psi(\lambda)$ denote the initial solution $I(\lambda), \varphi(\lambda)$ in terms of the variables J, ψ. Since $d\xi/d\lambda \to 0$ as $\lambda \to \pm\infty$, the substitution $I, \varphi \to J, \psi$ tends to become an indentity when $\lambda \to \pm\infty$. This implies that

$$J_\pm = \lim J(\lambda) \ \text{ when } \ \lambda \to \pm\infty, \ J_\pm = I_\pm.$$

exist. Now,

$$\Delta I = J_+ - J_- = -\int_{-\infty}^{\infty} \frac{\partial\Phi_1^{(N)}(J(\lambda), \psi(\lambda), \lambda)}{\partial\psi} d\lambda$$

By virtue of the Cauchy inequality we have, for real ψ,

$$\left|\frac{\partial\Phi_1^{(N)}}{\partial\psi}\right| \leqslant \frac{2}{\sigma_0} \sup_{|\mathrm{Im}\,\psi|<{}^1/_2\sigma_0} |\Phi_1^{(N)}|$$

and this yields

$$\Delta I = O(\|\Phi_1^{(N)}\|) = O(\exp(-c_1/\varepsilon))$$

Q.E.D.

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